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EXPONENTIAL STABILITY AND PATIENT RECOVERY DYNAMICS: MATHEMATICAL REHABILITATION FRAMEWORK

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Abstract: **Patient recovery is increasingly understood as a nonlinear adaptive process governed by biological, neurological, and therapeutic interactions. This paper develops a mathematically rigorous analysis for patient recovery using exponential stability theory, Lyapunov analysis, and nonlinear rehabilitation dynamics.**

Keywords: **Mathematical rehabilitation, exponential stability, dynamical systems, recovery dynamics.**

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1. INTRODUCTION

Recovery trajectories are nonlinear and adaptive processes influenced by neuroplasticity and rehabilitation mechanisms (Holt et al., 2010; Reinkensmeyer et al., 2016). Dynamical systems theory therefore provides a rigorous basis for understanding rehabilitation convergence.

Let patient functional state be given by:

$$r(t) \in R^n \qquad (1)$$

with dynamics $$\dot{r} = F(r, u, t) \qquad (2)$$

where: $r(t)$ denotes physiological state, and $u(t)$ denotes therapy input.

2. REVIEW OF RELEVANT LITERATURE:

2.1 Main Idea

Across rehabilitation science, computational neuroscience, and dynamic systems theory, a convergent theme emerges as patient recovery behaves as a dynamical system characterised by nonlinear adaptation, learning-driven change, and convergence toward stable functional states. Although authors differ in emphasis, some privileging mathematical models, others clinical trajectories or systems-level reorganisation—the literature consistently supports the idea that recovery follows predictable, stability-seeking dynamics. These dynamics often resemble exponential convergence, attractor formation, or constraint-driven re-stabilisation. Casadio and Sanguineti propose one of the most explicit mathematical models of recovery, conceptualising motor improvement during robot-assisted therapy as a learning system with retention and slacking. Their state-space formulation captures how patients adapt to errors, retain improvements, and sometimes reduce effort ("slacking") as assistance increases. The model predicts stable fixed points of performance, with therapy intensity modulating the rate of convergence toward these equilibria (Casadio & Sanguineti, 2012). This directly aligns with the concept of exponential stability, where system states converge toward a stable manifold under appropriate control inputs.

Liu et al. develop a biophysical dynamical model of muscle activation, fatigue, and recovery using differential equations. Their formulation explicitly models recovery as an exponential return to baseline, governed by time constants reflecting physiological processes (Liu et al., 2002). Although focused on muscle rather than neurological recovery, their work provides a mechanistic foundation for exponential stability in biological systems.

Reinkensmeyer and colleagues synthesise computational models of neuroplasticity, motor learning, and recovery, arguing that rehabilitation outcomes can be predicted by mechanistic models of plasticity and adaptation. Many of these models rely on exponential learning rules**,** Hebbian plasticity, or gradient-descent-like convergence, all of which inherently assume stable attractor states toward which behaviour evolves (Reinkensmeyer et al., 2016). Their work positions computational neurorehabilitation as a field grounded in stability-seeking dynamics. Krakauer frames stroke recovery as a form of motor learning, emphasising that learning curves typically follow exponential or power-law trajectories. He argues that recovery is constrained by time-sensitive windows of plasticity, producing predictable convergence toward functional stability (Krakauer, 2006). This perspective reinforces the idea that recovery is governed by learning-driven dynamical processes rather than purely biological healing.

3. EMPIRICAL PATTERNS OF FUNCTIONAL RECOVERY

Patterns of functional recovery after stroke. They describe a rapid early phase followed by progressive plateauing**,** a trajectory mathematically consistent with exponential saturation **(**Kwakkel et al., 2004). Initial impairment severity strongly predicts the stable endpoint, suggesting that recovery converges toward patient-specific attractor states**.**

In their comprehensive review, Langhorne et al. highlight the importance of dose, timing, and intensity **i**n shaping recovery trajectories. Although not explicitly mathematical, their synthesis assumes that recovery follows bounded, convergent patterns, with therapy modulating the rate but not the direction of convergence (Langhorne et al., 2011). This aligns with the broader stability framework.

Dombovy and colleagues provide empirical evidence from subarachnoid haemorrhage rehabilitation, showing that recovery follows nonlinear, stage-dependent trajectories**.** Patients exhibit rapid early gains followed by stabilisation, consistent with exponential-like decay toward a functional plateau (Dombovy et al., 1998). Although descriptive, their findings support the general dynamical pattern observed across neurological conditions.

Holt and colleagues apply dynamic systems theory to rehabilitation, emphasising self-organisation**,** constraints, and attractor landscapes. They argue that motor behaviour stabilises into attractor states shaped by task, organism, and environmental constraints (Holt et al., 2010). Recovery is conceptualised as movement between attractors, with therapy shifting system parameters to create new stable states. This framework highlights nonlinear stability, complementing exponential models.

Levac and DeMatteo extend dynamic systems theory to explain variability and nonlinearity in recovery. They emphasise phase shifts**,** emergent stability, and context-dependent attractor formation (Levac & DeMatteo, 2009). Their perspective challenges purely exponential models by highlighting multi-stability and nonlinear transitions, yet still situates recovery within a stability-seeking dynamical system.

4. CLINICAL GUIDELINES AND STABILITY ASSUMPTIONS

The American Heart Association/American Stroke Association guidelines synthesise evidence on stroke rehabilitation, emphasising task-specific training**,** intensity**,** and timing**.** While not mathematical, the guidelines implicitly assume that recovery follows predictable, convergent trajectories**,** with therapy influencing the slope of improvement rather than its ultimate stability (Winstein et al., 2016). This reflects a clinical consensus that recovery dynamics are stable and modifiable**.**

5. RECOVERY EQUILIBRIUM

Definition 1. From equation (2), *a recovery equilibrium* r^* *satisfies*

$$F\,(r^*, 0, t) = 0 \qquad (3)$$

Lemma 1. *Near equilibrium is*

$$\dot{y} = By \qquad (4)$$

where

$$B = \frac{\partial F}{\partial r}(r^*) \qquad (5)$$

Proof. Applying Taylor expansion we obtain,

$$F(r) = F(r^*) + B(r - r^*) + o(\|r - r^*\|)$$

Since $F(r^*) = 0$,

$$\dot{y} = By + o(\|y\|)$$

thus proving the result.

Definition 2. *A recovery equilibrium* r^* *is defined as a point satisfying*

$$F(r^*, 0, t) = 0. \qquad (8)$$

To see in full, how the linearisation near recovery equilibrium occurred,

Let

$$y = r - r^*.$$

Near the equilibrium r^*, *the dynamics are approximated by*

$$\dot{y} = By, \qquad (9)$$

Where

$$B = \frac{\partial F}{\partial r}(r^*, 0, t). \qquad (10)$$

Applying the Taylor expansion of $F(r,0,t)$ about r^* gives

$$\begin{aligned} F(r,0,t) &= F(r^*,0,t) + \frac{\partial F}{\partial r}(r^*,0,t)(r - r^*) \\ &\quad + \frac{1}{2!}D^2F(r^*,0,t)[r - r^*,\, r - r^*] \\ &\quad + \frac{1}{3!}D^3F(r^*,0,t)[r - r^*,\, r - r^*,\, r - r^*] + \cdots. \end{aligned} \qquad (11)$$

Since $F\ (r^*, 0, t)$= 0, substituting

$y = r - r^*$

yields

$$\begin{aligned} F(r,0,t) &= By + \frac{1}{2!}D^2F(r^*,0,t)[y,y] \\ &\quad + \frac{1}{3!}D^3F(r^*,0,t)[y,y,y] + \cdots \end{aligned} \qquad (12)$$

Therefore, the dynamics become

$$\dot{y} = By + \frac{1}{2!}D^2F(r^*,0,t)[y,y] + \frac{1}{3!}D^3F(r^*,0,t)[y,y,y] + \cdots \qquad (13)$$

Neglecting higher-order terms gives the linearized system

$$\dot{y} = By + o(\|y\|) \qquad (14)$$

□

Which is equation 7. For completeness, we shall consider the Taylor expansion in index notation in the next version of this paper.

6. MAIN RECOVERY THEOREM

Theorem 1. *Suppose all eigenvalues satisfy*

$$Re(\lambda_i(B)) < -\mu < 0 \qquad (15)$$

Then patient recovery converges exponentially toward equilibrium.

Proof. The linearized solution satisfies

$$y(t) = e^{Bt}y(0) \qquad (16)$$

Using semigroup estimates,

$$\|e^{Bt}\| \leq Ke^{-\mu t} \qquad (17)$$

Therefore,

$$\|r(t) - r^*\| \leq Ke^{-\mu t}\|r(0) - r^*\| \qquad (18)$$

Hence recovery converges exponentially. □

Corollary 1. *Larger spectral gaps correspond to faster rehabilitation rates and shorter recovery periods.*

7. LYAPUNOV REHABILITATION ANALYSIS

Define the Lyapunov function

$$W(r) = r^TQr \qquad (19)$$

where $Q > 0$.

Theorem 2. *Suppose*

$$\dot{W} \leq -Hw \tag{20}$$

Then the rehabilitation equilibrium is globally exponentially stable.

Proof. Integrating,

$$W(t) \leq W(0)e^{-\eta t}$$

By positive definiteness and radial unboundedness, (21)

$$\|r(t) - r^*\| \to 0 \tag{22}$$

exponentially as $t \to \infty$.

exponentially as $t \to \infty$. ☐

8. CLINICAL INTERPRETATION

Recovery exponents may therefore function as quantitative biomarkers of therapeutic

Efficiency, so that the framework captures clinical issues relating to the under listed:

(i) stroke recovery,

(ii) motor relearning,

(iii) neuroplasticity adaptation,

(iv) ICU rehabilitation,

(v) fatigue-recovery cycling

9. SYNTHESIS AND THEORETICAL INTEGRATION

Across this work three unifying principles emerge:

(i) Recovery is a Dynamical System

Whether framed through computational models (Casadio & Sanguineti; Reinkensmeyer et al.), motor learning (Krakauer), or dynamic systems theory (Holt et al.; Levac & DeMatteo), recovery is consistently described as state-dependent, adaptive, and stability-seeking**.**

(ii) Exponential-Like Convergence is Common

Empirical studies (Kwakkel et al.; Dombovy et al.) and mechanistic models (Liu et al.) show that recovery often follows rapid early gains followed by slowing improvement**,** mathematically analogous to exponential convergence toward a stable equilibrium.

(iii) Nonlinearities and Attractors Shape Recovery

Dynamic systems authors emphasise multiple attractors**,** phase transitions, and constraint-driven reorganisation, suggesting that recovery is not always smooth but remains fundamentally stability-oriented**.**

10. CONCLUSION

The literature collectively supports a robust theoretical position: patient recovery can be modelled as a dynamical system exhibiting exponential stability toward functional attractors**.** Mathematical models provide mechanistic clarity, empirical studies confirm predictable convergence patterns, and dynamic systems theory explains nonlinear transitions and multi-stability. Together, these perspectives form a coherent framework for a mathematical rehabilitation model in which recovery dynamics are:

(i) Predictable

(ii) Stability-seeking

(iii) Modifiable through therapy

(iv) Constrained by neurobiological and behavioural parameters

This synthesis provides a strong conceptual foundation for developing formal dynamical models of rehabilitation, integrating exponential stability with nonlinear systems theory to better predict and optimise patient recovery.

Exponential stability theory provides a mathematically rigorous basis for rehabilitation science and patient recovery analysis. Recovery may be interpreted as convergence toward physiological equilibrium governed by nonlinear adaptive dynamics.

Declaration: All Authors declare that they read final version of manuscript of this paper, and confirm that there is no conflict of interest.

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